Modular Resonance in the Divisor Function: Structural Behavior of $\sigma(n)/n$ Across Periodic Grids

Troy Winarski

April 20, 2025

Abstract

We investigate the behavior of the normalized divisor function $\sigma(n)/n$ when integers are arranged into a modular grid defined by a fixed period p. Each column consists of values spaced by p, and rows are formed by sampling horizontally across columns. Our analysis shows that when p is prime, the row-wise averages of $\sigma(n)/n$ remain strikingly flat. In contrast, composite periods introduce oscillatory deviations aligned with their factor structure. Despite these local variations, the global average of $\sigma(n)/n$ across the dataset converges to the classical value $\pi^2/6$, as established in analytic number theory by Euler and Dirichlet. This paper establishes the structure of the function under modular binning and provides a reproducible framework—visual and numerical—for detecting and classifying periodic resonance in multiplicative arithmetic functions.

1. Introduction

The divisor function $\sigma(n)$ gives the sum of all positive divisors of n. When normalized by n, the ratio $\sigma(n)/n$ exhibits an average value known to converge to $\pi^2/6$ over the natural numbers. In this paper, we investigate how this function behaves under modular binning: data is structured into a two-dimensional grid where each column contains values of the form $c + kp$, with p being a fixed period. Each row of the grid samples horizontally across the columns, yielding sets of the form $\{r + 0 \cdot p, r + 1 \cdot p, \dots\}$.

We compute the average of $\sigma(n)/n$ across each row and observe how these averages change depending on whether p is prime or composite. This setup transforms a one-dimensional arithmetic sequence into a modular resonance field, revealing structural regularities not visible through direct examination.

2. Methodology

We construct a grid of height p, meaning each column contains p integers. The number of columns is determined by the total sample size. For a given sample, we compute:

- **Row index** r: determines the starting value of the row.

- **Row elements**: $\{r + kp \mid 0 \leq k < C\}$, where C is the number of columns.
- **Row average**: $R_{\sigma/n}(r;p) = \frac{1}{C}\sum_{k=0}^{C-1}\frac{\sigma(r+kp)}{r+kp}$

We evaluate this structure for various values of p, comparing prime and composite periods. For meaningful stability, we use a minimum of 10,000 values in each grid, balancing clarity and performance.

3. Empirical Observations

Our key findings include:

- When p is **prime**, the row averages remain nearly constant across all r, resulting in a flat average profile, which is represented as a vertical bar chart in the accompanying figures.
- When p is **composite**, the row averages form visible wave patterns that correlate with the factor structure of p.
- Despite local variability, the overall average of all $\sigma(n)/n$ values within the sample still converges closely to $\pi^2/6$.

When p is prime, one row in the modular grid—specifically, the row containing only non-primes—tends to exhibit a visibly higher average. This occurs because all other rows contain a mix of composite and prime values, with primes lowering the local average due to the relatively small size of their divisor sums. Since the global average must still converge to $\pi^2/6$, the elevated value in the prime-free row imposes a slight reduction across all other rows to compensate. This distribution of "excess weight" reinforces the underlying modular balance and highlights the unique role of prime sparsity in shaping local average behavior.

4. Conclusion

The modular averaging of $\sigma(n)/n$ exposes a resonant framework within number theory. Prime periods induce modular equilibrium, while composite periods resonate with their internal structure. This approach paves the way for deeper explorations into modular symmetries in arithmetic functions.

Q.E.D.

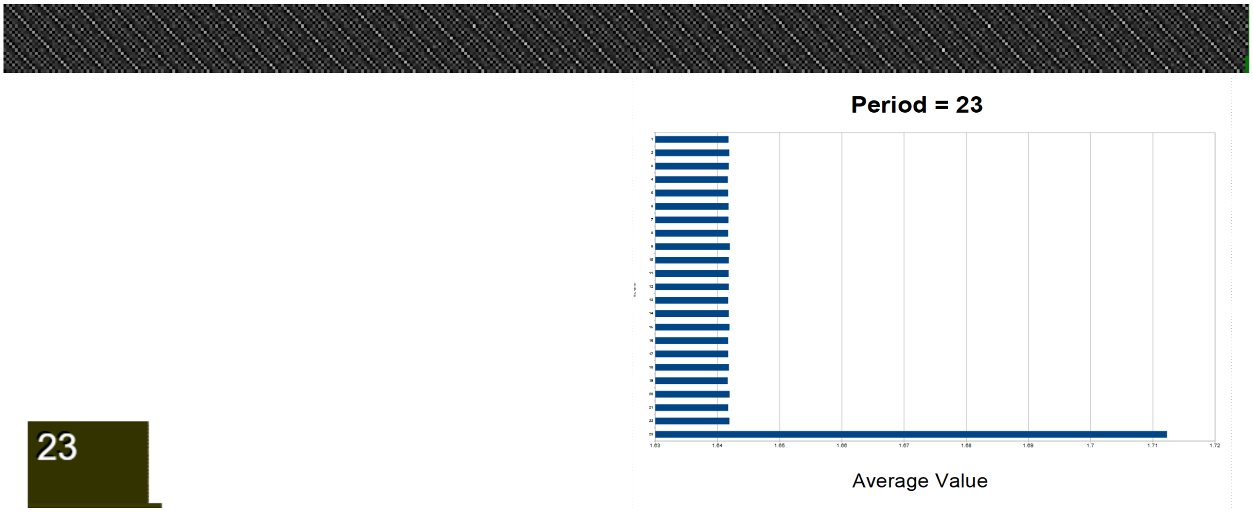

Figure 1: Period $p = 23$. Top: Modular grid of $\sigma(n)/n$ (10,000 samples). Bottom left: Row-wise averages (100,000 samples). Bottom right: Refined averages (100,000 samples).

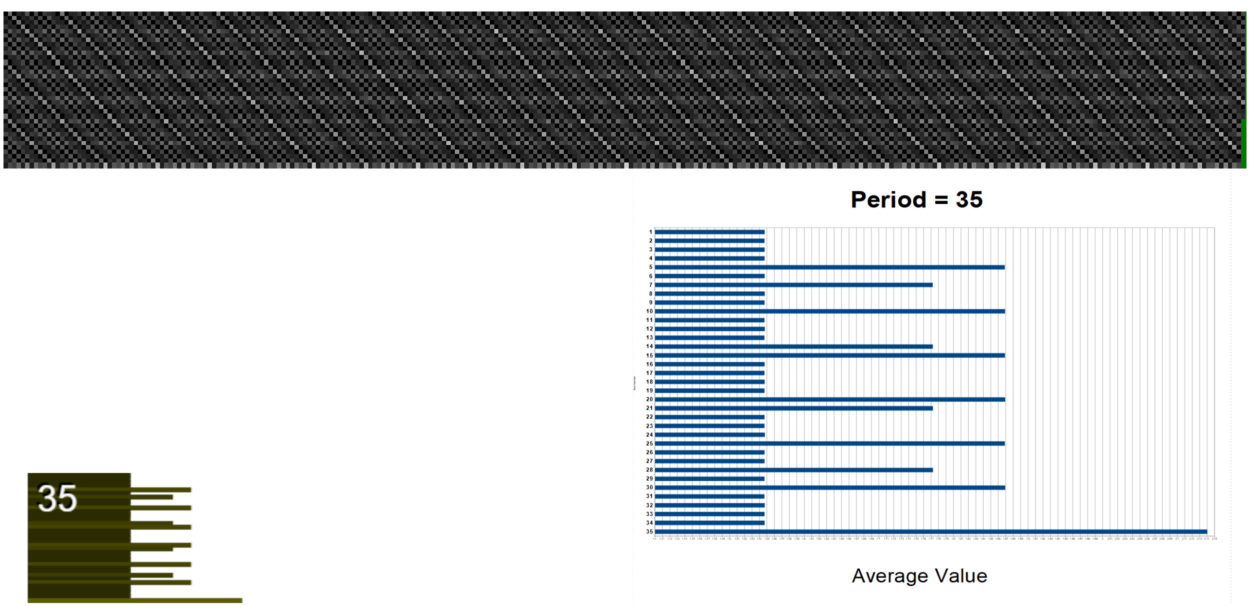

Figure 2: Period $p = 35$. Top: Modular grid of $\sigma(n)/n$ (10,000 samples). Bottom left: Row-wise averages (100,000 samples). Bottom right: Refined averages (100,000 samples).

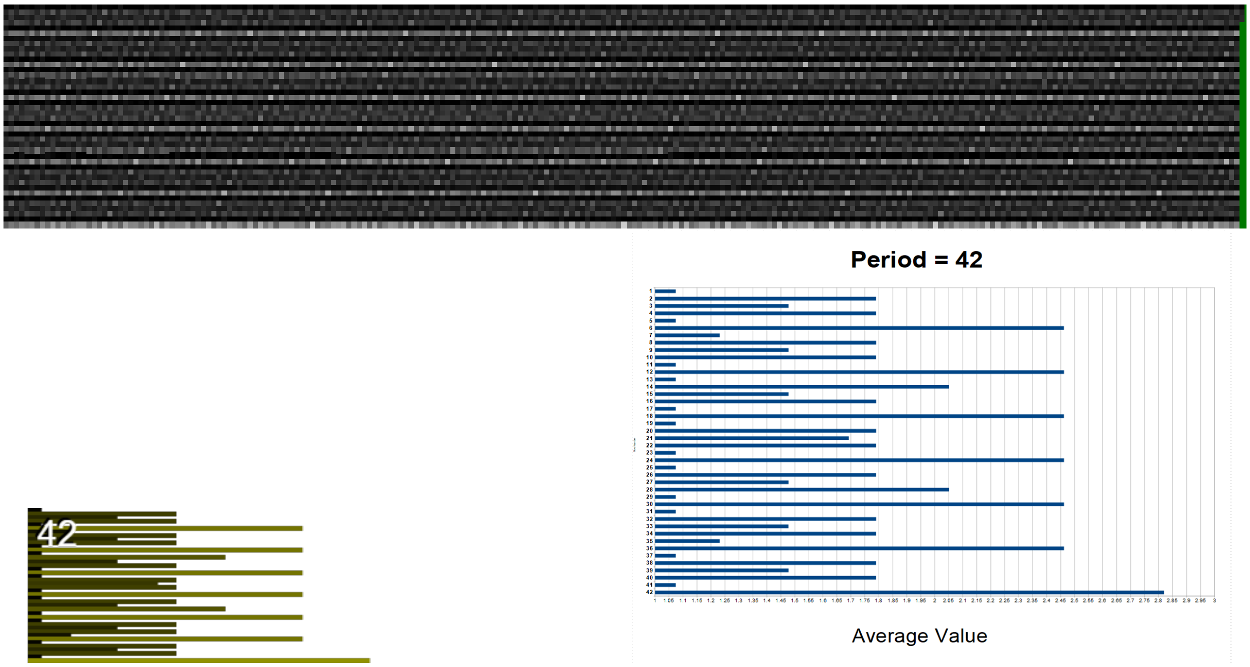

Figure 3: Period $p = 42$. Top: Modular grid of $\sigma(n)/n$ (10,000 samples). Bottom left: Row-wise averages (100,000 samples). Bottom right: Refined averages (100,000 samples).

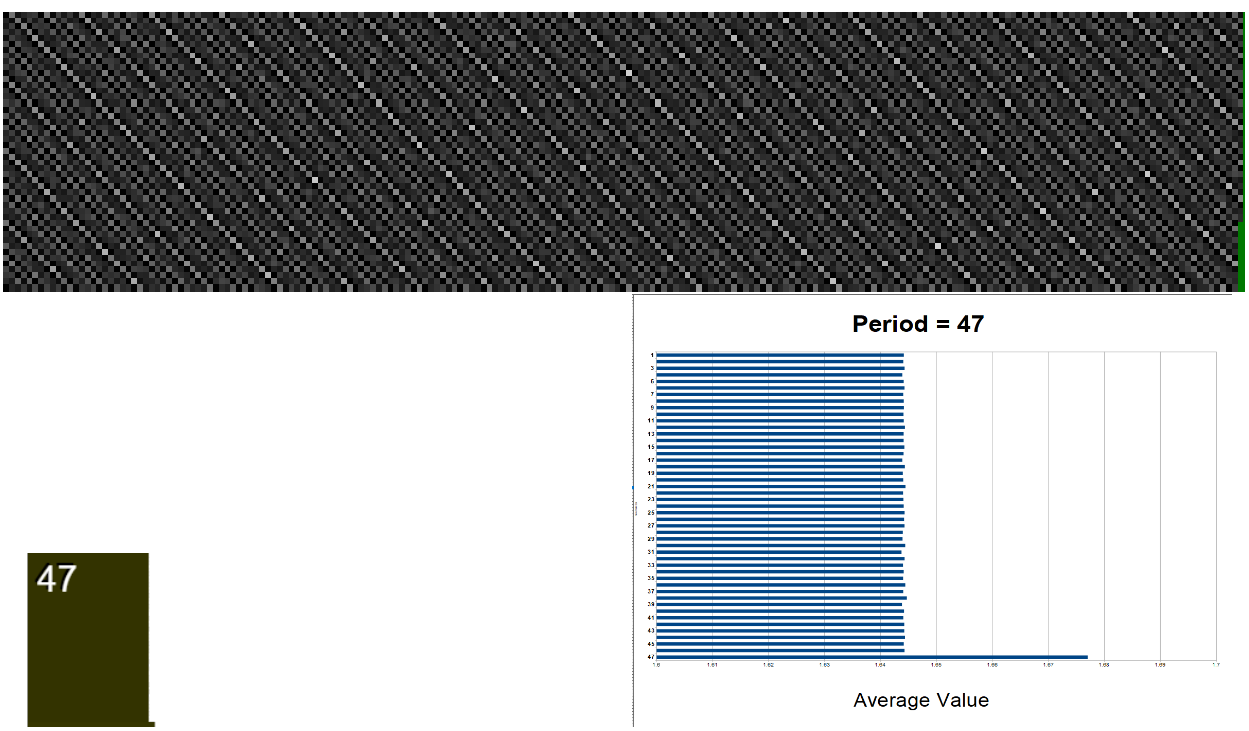

Figure 4: Period $p = 47$. Top: Modular grid of $\sigma(n)/n$ (10,000 samples). Bottom left: Row-wise averages (100,000 samples). Bottom right: Refined averages (100,000 samples).

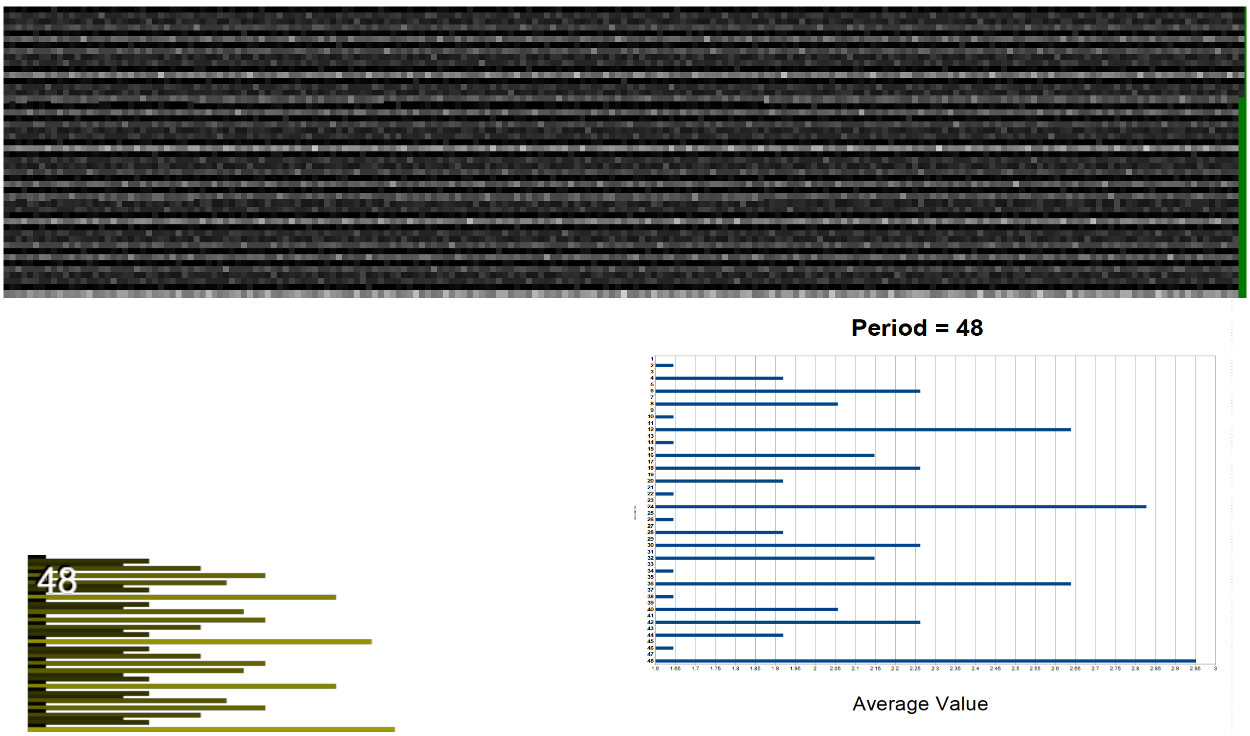

Figure 5: Period $p = 48$. Top: Modular grid of $\sigma(n)/n$ (10,000 samples). Bottom left: Row-wise averages (100,000 samples). Bottom right: Refined averages (100,000 samples).

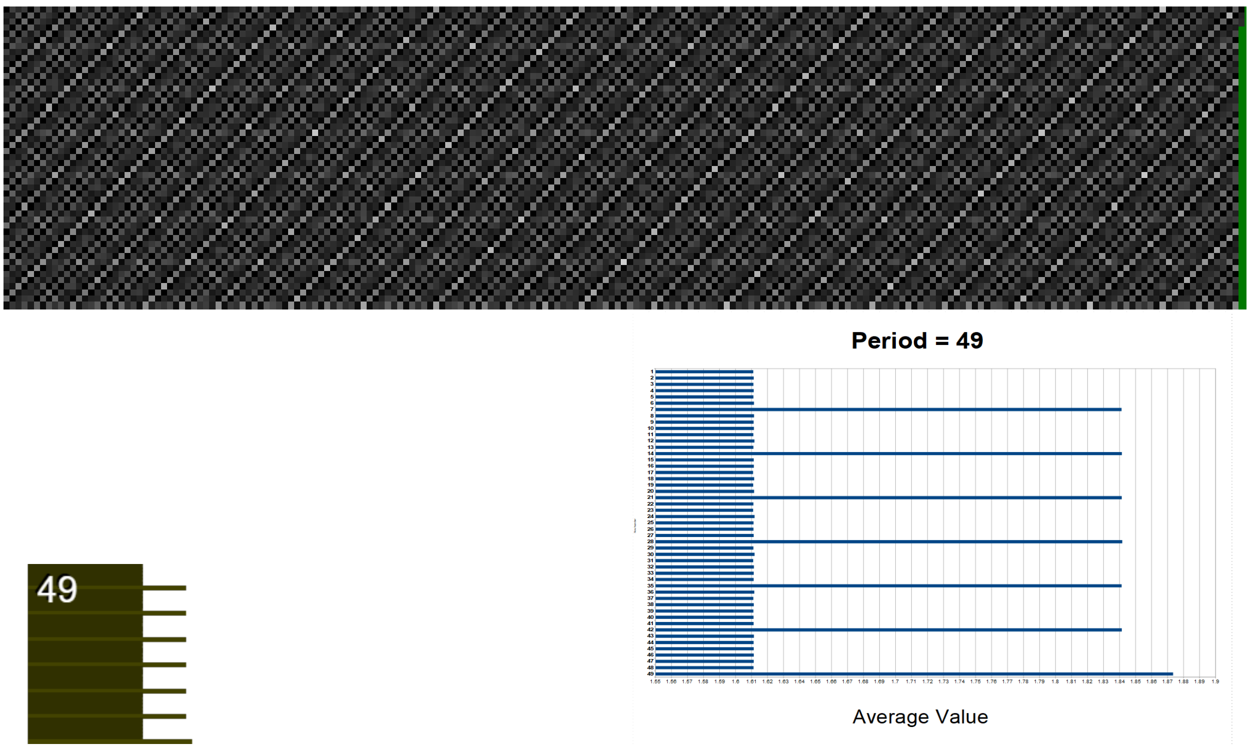

Figure 6: Period $p = 49$. Top: Modular grid of $\sigma(n)/n$ (10,000 samples). Bottom left: Row-wise averages (100,000 samples). Bottom right: Refined averages (100,000 samples).

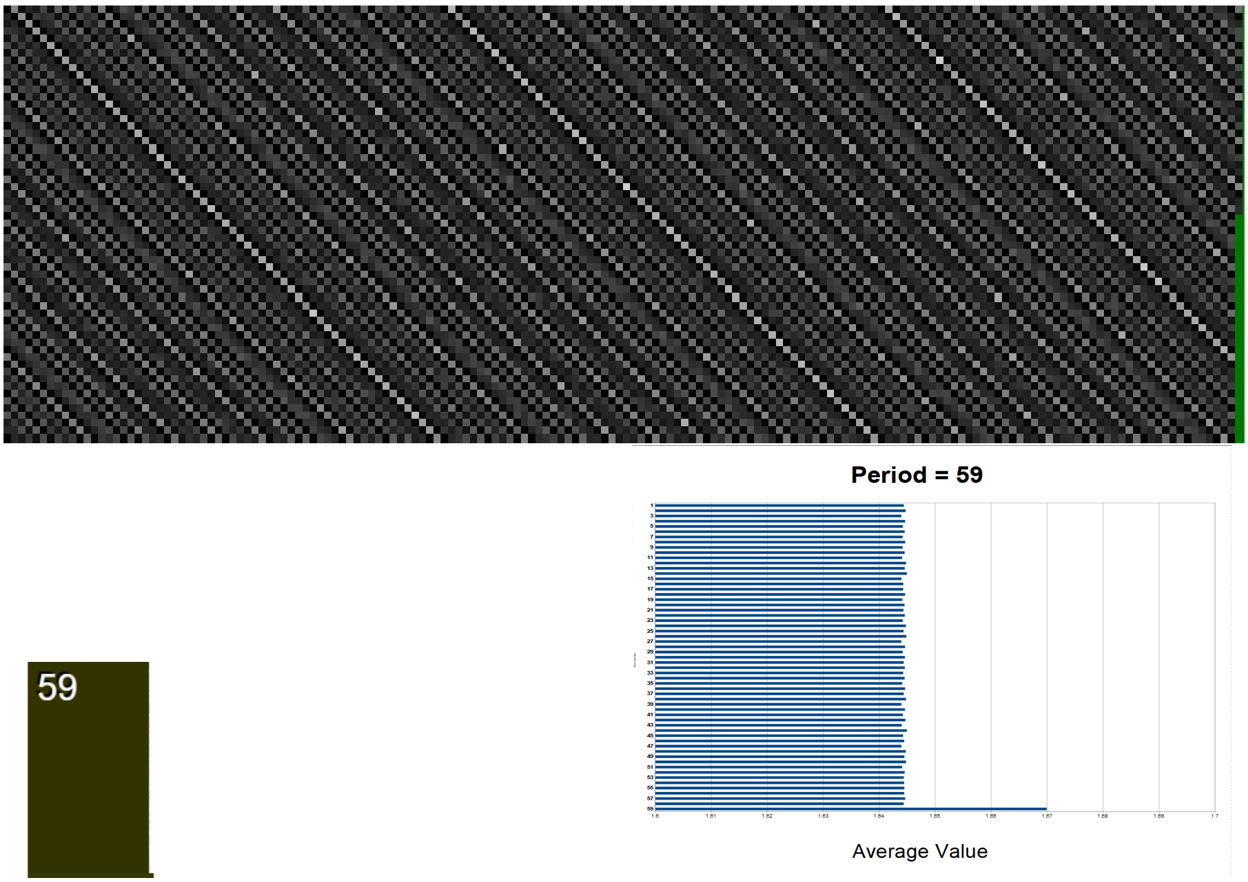

Figure 7: Period $p = 59$. Top: Modular grid of $\sigma(n)/n$ (10,000 samples). Bottom left: Row-wise averages (100,000 samples). Bottom right: Refined averages (100,000 samples).

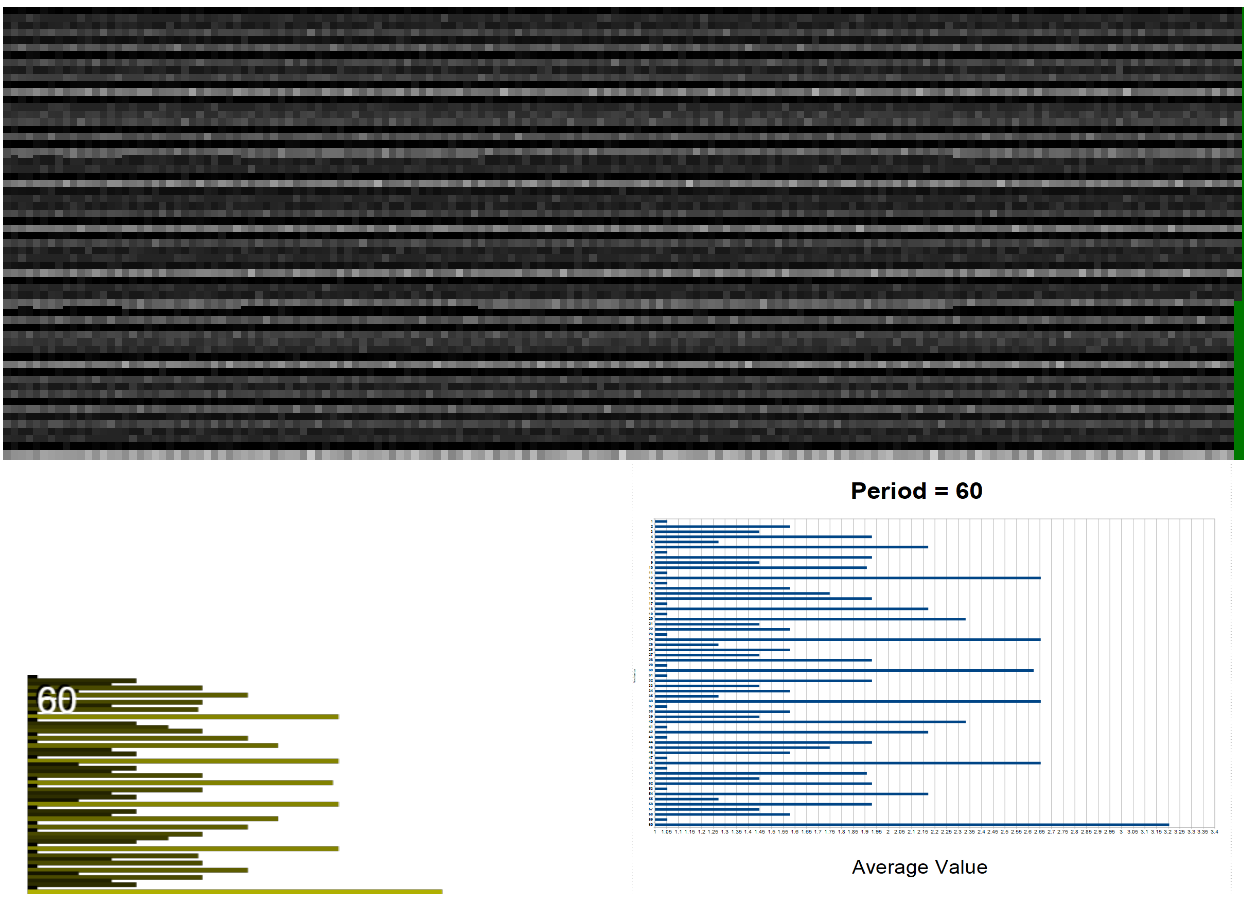

Figure 8: Period $p = 60$. Top: Modular grid of $\sigma(n)/n$ (10,000 samples). Bottom left: Row-wise averages (100,000 samples). Bottom right: Refined averages (100,000 samples).

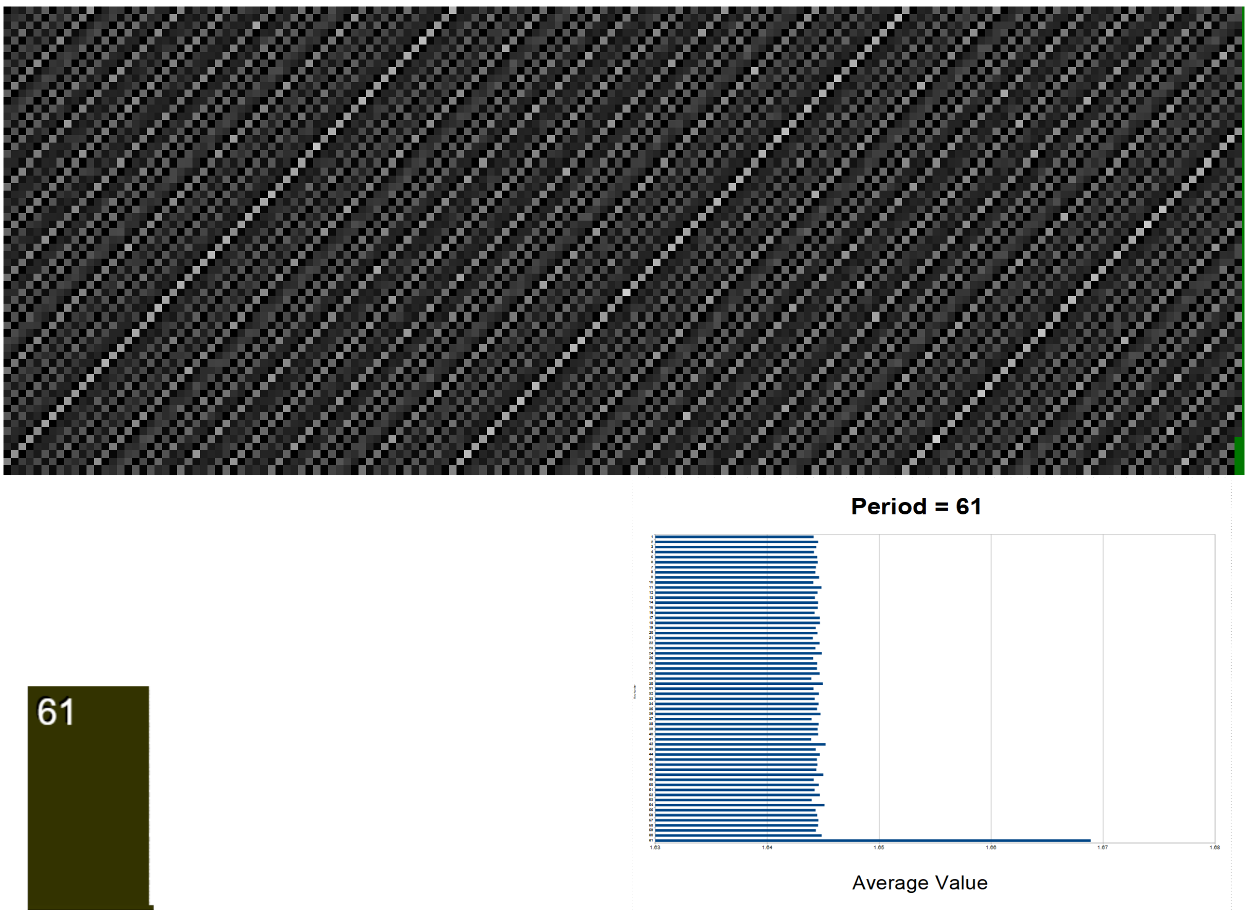

Figure 9: Period $p = 61$. Top: Modular grid of $\sigma(n)/n$ (10,000 samples). Bottom left: Row-wise averages (100,000 samples). Bottom right: Refined averages (100,000 samples).

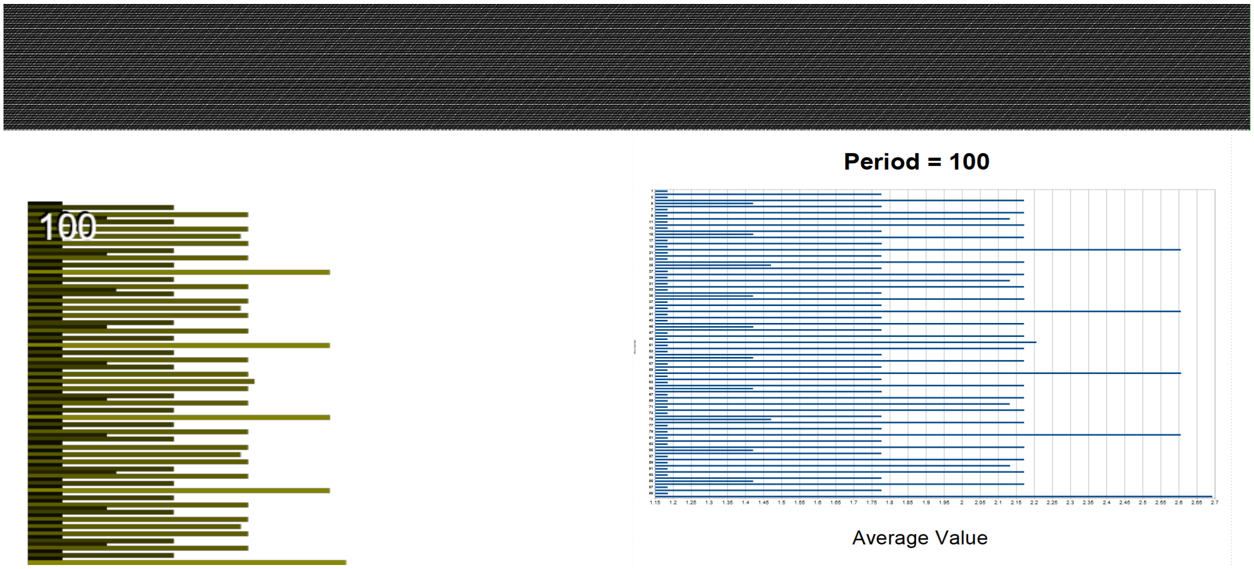

Figure 10: Period $p = 100$. Top: Modular grid of $\sigma(n)/n$. Bottom left: Row-wise averages (100,000 samples). Bottom right: Refined averages (100,000 samples).

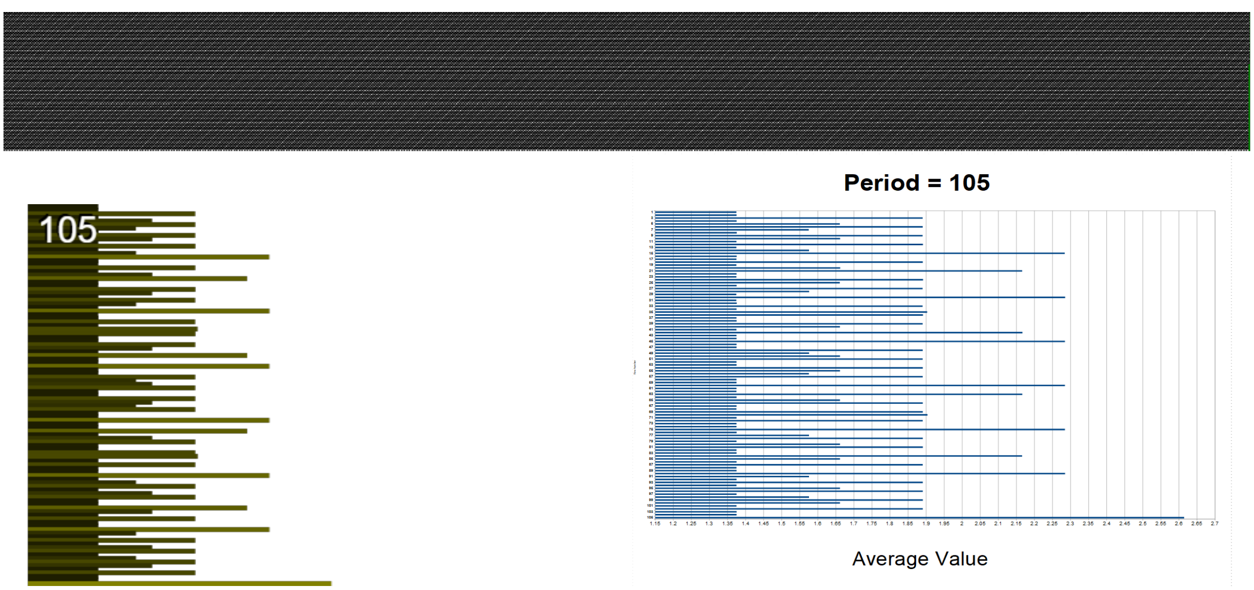

Figure 11: Period $p = 105$. Top: Modular grid of $\sigma(n)/n$. Bottom left: Row-wise averages (100,000 samples). Bottom right: Refined averages (100,000 samples).

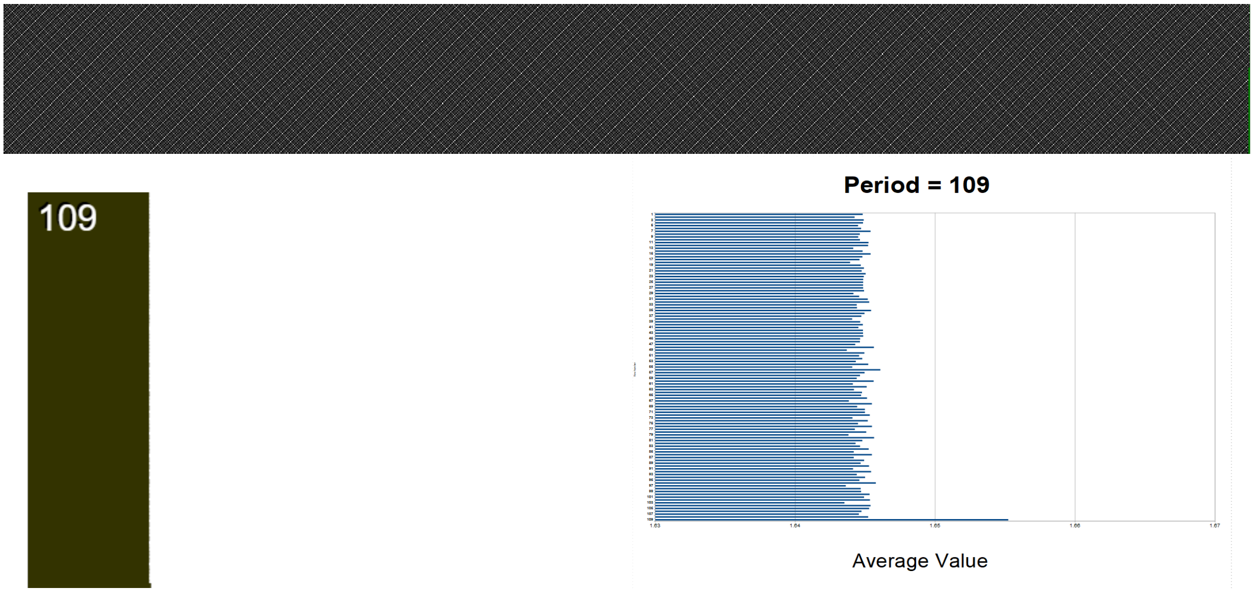

Figure 12: Period $p = 109$. Top: Modular grid of $\sigma(n)/n$. Bottom left: Row-wise averages (100,000 samples). Bottom right: Refined averages (100,000 samples).

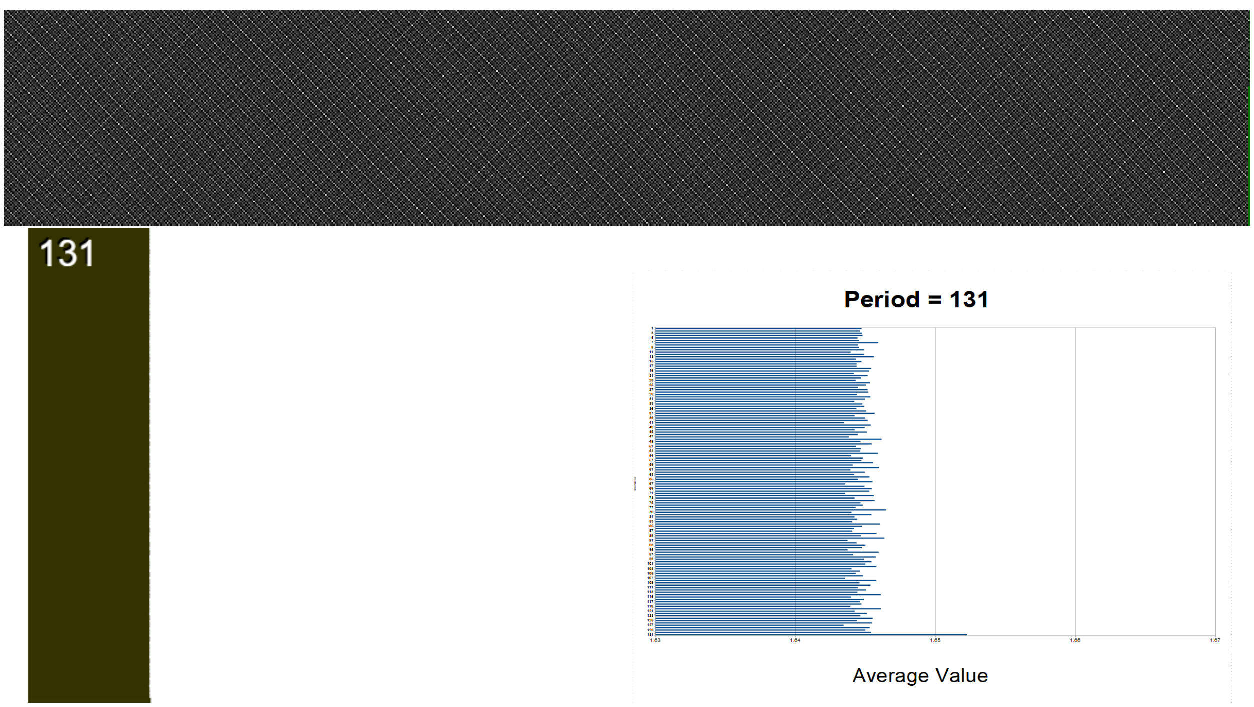

Figure 13: Period $p = 131$. Top: Modular grid of $\sigma(n)/n$. Bottom left: Row-wise averages (100,000 samples). Bottom right: Refined averages (100,000 samples).

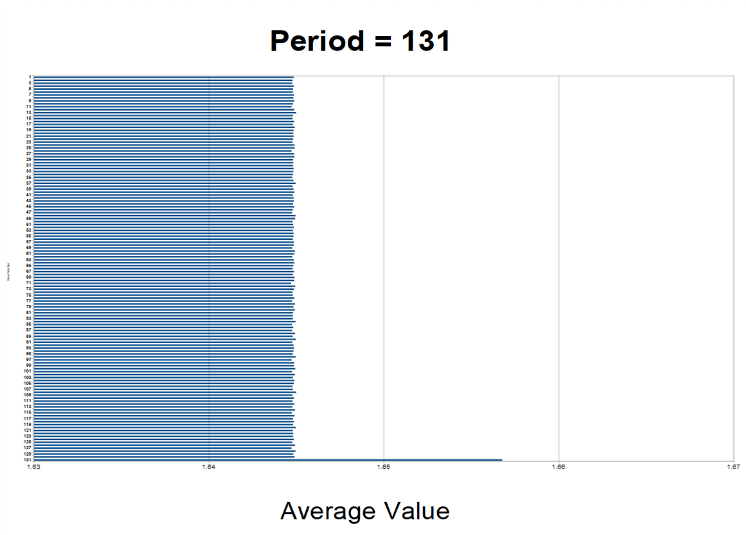

Figure 14: Refined row average profile for period $p = 131$ using 1,000,000 values. The increased data sample visibly reduces variance and approaches the nominal average structure.